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LETTER TO THE EDITOR

The quantum Jeffreys' prior/Bures metric volume element for squeezed thermal states and a universal coding conjecture

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Abstract. Clarke and Barron have recently established that the (classical) Jeffreys' prior yields universal codes—ones which do relatively well (in the sense of relative entropy) no matter what the true state. Twamley has recently computed the Bures metric—proportional to the quantum Fisher information (statistical distinguishability) metric, as shown by Braunstein and Caves—for squeezed thermal states. The volume elements of these metrics—that is, the quantum Jeffreys' prior—are found to be simply the product of a function of the squeeze factor and a function of the inverse temperature, the phase being irrelevant in this regard. A computational strategy to find a universal coding using the quantum Jeffreys' prior—previously implemented for the two-level quantum systems—is then discussed.

The (classical) Jeffreys' prior—widely used in Bayesian analyses—is defined as the square root of the determinant of the $d \times d$ Fisher information matrix of a d-variate probability distribution [1]. This matrix is the negative of the expected value of the Hessian of the logarithm of the probability density. Recently [2, 3], in analogy to this definition, we have considered the quantum Jeffreys' prior to be the square root of the determinant of the $d \times d$ quantum Fisher information matrix [4–6] for a d-parameter set of density matrices. Here, the symmetrized logarithmic derivative is used to compute the information matrix, due to the general non-commutativity of observables in quantum mechanics.

The quantum Jeffreys' prior has been found [2, 3] and normalized to a (prior) probability distribution over: (1) the three-dimensional convex set of 2×2 (complex) density matrices; (2) the five-dimensional convex set of 2×2 (quaternionic) density matrices; and (3) a four-dimensional convex set of 3×3 density matrices, reducing to (1) for a fixed value of a specific one of the four parameters. We have also computed [3] the 8×8 quantum Fisher information matrix for the eight-dimensional convex set of three-level (spin-1) density matrices—using the parametrization suggested in [7]—but have been unable to compute its determinant, without fixing at least four of the eight parameters.

In any case, however, it is possible—relying upon the recently demonstrated [8] simple proportionality between the quantum Fisher information metric and the (natural) Bures metric—to take the quantum Jeffreys' prior for a class of $n \times n$ density matrices (ρ) to be proportional to the product of $|\rho|^{-1/2}$ and

$$\prod_{1 \leqslant i < j \leqslant n} 1/(\lambda_i + \lambda_j) \tag{1}$$

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where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of ρ. The product of (1) and $|\rho|^{-1/2}$ is then the volume element of the Bures metric. This follows from the formula from Hübner [9] for the Bures metric:

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\langle i|\mathrm{d}\rho|j\rangle|^2}{\lambda_i+\lambda_j} \tag{2}$$

by writing the metric in terms of a set of independent differentials ($n-1$ of them for the diagonal entries and $n(n-1)$ for the off-diagonal entries). Equation (1) can be expressed either in terms of the complete (h) or elementary (e) symmetric functions of the n eigenvalues by calculating the determinant of an $(n-1)\times(n-1)$ matrix, the ij-entries of which are either h_{2i-j} or e_{2i-j}. (This result, as applied to the h's is given in [10] (p 29, exercise 7). Ira Gessel (personal communication) has indicated that it must also hold for the e's since (1) is the Schur function $s_{(n-1,n-2,\ldots,1)}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(n-1, n-2, \ldots, 1)$ is a self-conjugate partition [10, (3.5)]. One can then compute (1) from the characteristic polynomial of ρ [11], without having to directly determine the individual eigenvalues (cf [9]). Normalization, however, of the quantum Jeffreys' prior over high-dimensional convex sets may prove problematical.

An asymptotic (minimax relative entropy) property of the (classical) Jeffreys' prior has recently been established [12, 13; cf 14]. We have sought to extend these results to the quantum domain, starting with the case of the two-level (complex) systems [15]. Computations have been performed that are supportive (but not yet probative) of the following proposition. If one averages the $2^n \times 2^n$ density matrices $\overset{n}{\otimes}\rho$ over the three-dimensional convex set (B) of 2×2 density matrices (ρ), then if the averaging is performed using the quantum Jeffreys' prior over B, the maximum over B of the relative entropy of $\overset{n}{\otimes}\rho$ ($n\to\infty$) with respect to the $2^n\times 2^n$ averaged density matrix is minimized. In this sense, Jeffreys' prior would provide a 'universal coding' [12–14] of the two-level systems. This minimax relative entropy then was conjectured to approach $\frac{1}{2}\log\pi - 2\log 2 + \frac{3}{2}\log n$ as $n\to\infty$ [15].

The specific purpose of this letter is to report the quantum Jeffreys' prior and its properties for another class of density matrices—those for the (undisplaced) squeezed thermal states. We rely upon recent results of Twamley [16]. These density matrices are parametrizable in the form

$$\rho(\beta, r, \theta) = ZS(r,\theta)T(\beta)S^{+}(r,\theta) \qquad (0\leqslant\beta; 0\leqslant r; -\pi<\theta\leqslant\pi) \tag{3}$$

where

$$\begin{aligned} S(r,\theta) &= \exp(\zeta K_+ - \zeta^* K_-)\\ T(\beta) &= \exp(-\beta K_0)\\ \zeta &= r\,\mathrm{e}^{\mathrm{i}\theta} \end{aligned} \tag{4}$$

and

$$\begin{aligned} &K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \qquad K_0 = \tfrac{1}{2}(a^\dagger a + \tfrac{1}{2})\\ &[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0. \end{aligned} \tag{5}$$

Here $S(r,\theta)$ is the one-photon squeeze operator, a is the single mode annihilation operator, Z is chosen so that $\mathrm{Tr}(\rho)=1$, and (K_0, K_+) are the generators of the $SU(1,1)$ group.

The Bures metric can then be expressed—in diagonal form [17]—as either [16]

$$\frac{1}{2}[1+\mathrm{sech}\,\beta/2](\mathrm{d}r^2+\sinh^2(2r)\,\mathrm{d}\theta^2)+\frac{1}{64\sinh^2\beta/4}\,\mathrm{d}\beta^2 \tag{6}$$

or, defining $\exp(-2u) = \tanh\beta/8$,

$$\tfrac{1}{2}[1 + \tanh^2 u](\mathrm{d}r^2 + \sinh^2(2r)\,\mathrm{d}\theta^2) + \mathrm{d}u^2. \tag{7}$$

The volume element (proportional to the quantum Jeffreys' prior) associated with (6) is of the product form $f(r)g(\beta)$, where

$$f(r) = \sinh 2r \tag{8}$$

and

$$g(\beta) = (\cosh\beta/4 \coth\beta/4 \operatorname{sech}\beta/2)/8 \tag{9}$$

while for (7), the prior is of the form $f(r)h(u)$, where

$$h(u) = (\cosh 2u \operatorname{sech}^2 u)/2. \tag{10}$$

The univariate function $f(r)$ can be normalized over the range $r \in [0, R]$ by dividing by

$$\sinh^2 R \tag{11}$$

while the univariate function $h(u)$ can be normalized over the range $u \in [0, U]$ by dividing by

$$U - (\tanh U)/2. \tag{12}$$

In figure 1 is shown $f(r)$ normalized over the range $[0, 5]$ and in figure 2, $h(u)$, normalized over $[0, 5]$. In figure 3 is shown $g(\beta)$, over the range $\beta \in [0, 5]$. (We have not succeeded in normalizing—through exact nor numerical integration—$g(\beta)$ over ranges of the form $[0, B]$).

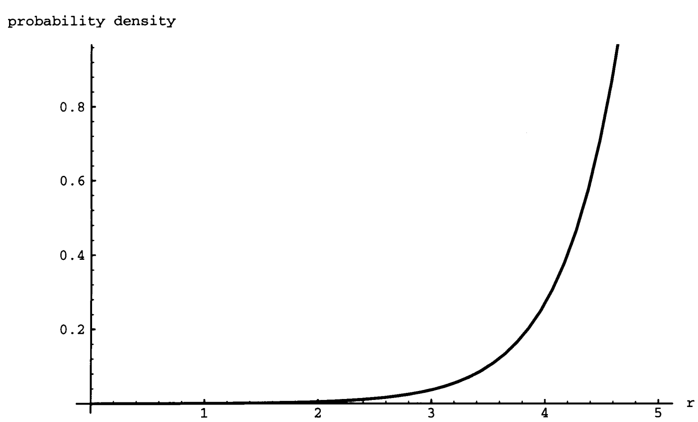

Figure 1. Univariate marginal ($f(r)$) of quantum Jeffeys' prior normalized over $r \in [0, 5]$.

A phenomenonologically meaningful choice for the squeezing parameter cutoff (R) might be the critical value [18]

$$r_s = \tfrac{1}{2}\log(2\bar{n} + 1) \tag{13}$$

where

$$\bar{n} = [\exp[\omega\beta] - 1]^{-1} \tag{14}$$

is the mean occupancy and ω, the angular frequency of a single-mode radiation field. For $r > r_s$, pairwise oscillations of the photon-number distribution arise [18].

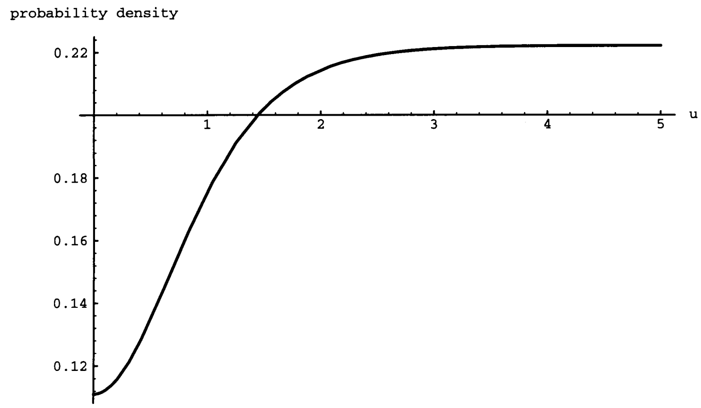

Figure 2. Univariate marginal ($h(u)$) of quantum Jeffreys' prior normalized over $u \in [0, 5]$.

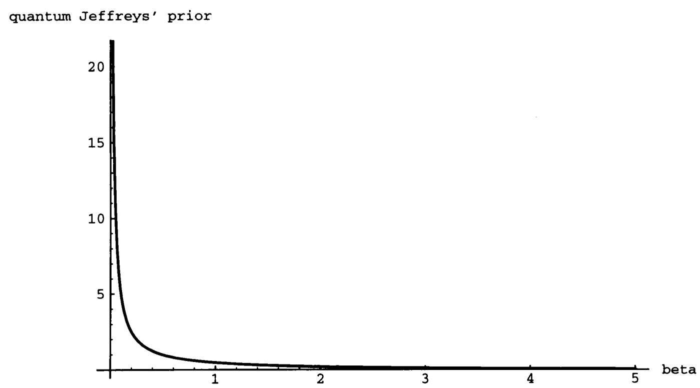

Figure 3. Unnormalized univariate marginal ($g(\beta)$) of quantum Jeffreys' prior over $\beta \in [0, 5]$.

Marian and Marian [19] have derived, besides, the usual quasiprobability densities, the coherent-state, number-state, coordinate and momentum representations of the density operator of squeezed states with thermal noise. Using such representations, we intend to conduct analyses parallel to those reported in [15]. There, n-fold tensor products ($n = 2, 3, 4$) $\overset{n}{\otimes}\rho$ of the two-level density matrices (ρ) were averaged over the three-dimensional convex set (B) of all such possible density matrices, using a one-parameter family of probability distributions, $q(u)$, $-\infty < u < 1$, for which $q(\frac{1}{2})$ is the (normalized) quantum Jeffreys' prior. The value of $u = U_n$ was found for which the maximum over B of the relative entropy of $\overset{n}{\otimes}\rho$ with respect to $2^n \times 2^n$ density matrices, $\zeta_n(u)$, was minimized. The matrices $\zeta_n(u)$ were obtained by averaging the $\overset{n}{\otimes}\rho$'s, using the probability distributions $q(u)$. It was found that $U_2 \approx 0.992$, $U_3 \approx 0.952$ and $U_4 \approx 0.912$. This decreasing trend is not inconsistent with the hypothesis—extending the results of Clarke and Barron [12, 13] to the quantum domain—that $\lim_{n\to\infty} U_n = \frac{1}{2}$. Under such a hypothesis, it was argued in [15], on the basis of the combinatorial structures observed, that the minimax relative entropy or risk would, asymptotically, approach $\frac{1}{2}\log\pi - 2\log 2 + \frac{3}{2}\log n$. The (non-quantum) result of Clarke and Barron [12, 13] is that the 'minimax risk' assumes the form

$$\log \int_K |I(\theta)|^{1/2}\,\mathrm{d}\theta + \frac{d}{2}\log\frac{n}{2\pi\mathrm{e}}. \tag{15}$$

Here, θ is a d-dimensional vector of variables parametrizing a family of probability distributions, $I(\theta)$ is the $d \times d$ Fisher information matrix and K is a compact set in the interior of the domain of the parameters.

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Note added in proof. C Krattenhaler has found the asymptotic form of the relative entropy of $\overset{n}{\otimes}\rho$ with respect to $\zeta_n(u)$. For $u = 1/2$ (the quantum Jeffreys' prior), it is

$$\frac{3}{2}(\log n - \log 2) - \frac{1}{2} + \frac{1}{2}\log\pi - \frac{1}{2}\log(1-r^2) + \frac{1}{2r}\log\left(\frac{1-r}{1+r}\right) + O(1/n)$$

where r is the distance of ρ from the origin in the Bloch sphere representation of the two-level systems.

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